Infinite Series

Convergence & Divergence Tests

Let $\sum a_k$ and $\sum b_k$ be series.

Test	Statement of Theorem	Application Notes
Divergence Test	If $\lim_{k\to\infty} a_k \neq 0$ then the series diverges. The converse is not necessarily true.	Shows only divergence.
Integral Test	If f is a function that is *continuous*, *decreasing*, and *positive* on an interval $[a,+\infty)$ such that and for all $k \geq a$ $a_k = f(k)$, then $\sum_{k=a}^{\infty} a_k$ and $\int_a^{+\infty} f(x)dx$ both converge, or both diverge.	Use if $f(x)$ is easy to integrate. Does *not* mean integral and series have same value.
Direct Comparison Test	Let $\sum a_k$ and $\sum b_k$ be series with *nonnegative terms*. If $a_k \leq b_k$ for all k greater than some number, then (a) If $\sum a_k$ diverges both series diverge. (b) If $\sum b_k$ converges both series converge.	Harder to apply than other tests. Requires a suitable comparison series to be chosen.
Limit Comparison Test	Given $\sum a_k$ and $\sum b_k$ with *nonnegative terms*, if we define $\rho \triangleq \lim_{k\to\infty} \frac{a_k}{b_k}$ then, (a) If $\rho \in (0,+\infty)$ both series converge, or both diverge. (b) If $\rho = 0$ and $\sum b_k$ converges then $\sum a_k$ converges. (c) If $\rho = +\infty$ and $\sum b_k$ diverges then $\sum a_k$ diverges.	Easier to apply than the Direct Comparison Test, but still requires a suitable comparison series to be chosen.
Ratio Test	If the limit of the ratio of successive terms of a series $\sum a_k$ with *nonnegative terms* is defined as $\rho \triangleq \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ then, (a) If $\rho < 1$ the series converges. (b) If $\rho > 1$ the series diverges. (c) If $\rho = 1$ the test is inconclusive.	Use when terms include factorials or k^{th} powers.
Root Test	If the limit of the k^{th} root of the k^{th} term of a series $\sum a_k$ with *nonnegative terms* is defined as $\rho \triangleq \lim_{k\to\infty} a_k^{1/k}$ then, (a) If $\rho < 1$ the series converges. (b) If $\rho > 1$ the series diverges. (c) If $\rho = 1$ the test is inconclusive.	Use when terms include k^{th} powers.
Absolute Convergence	A series $\sum a_k$ converges absolutely if the series $\sum \lvert a_k \rvert$ converges.	A series either: diverges, converges conditionally or converges absolutely.
Alternating Series Test	The alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k > 0$, converges if the following two conditions are met (*remember that a_k is positive*) (a) $a_k \geq a_{k+1}$ for all k. (eventually) ((strictly) monotonically decreasing) (b) $\lim_{k\to\infty} a_k = 0$	Only applicable to alternating series.

Summary of Infinite Series & Limits

Series	Convergence/Divergence
Geometric Series	$\sum_{k=0}^{\infty} ar^k = \begin{cases} \frac{a}{(1-r)} & \text{for } \lvert r \rvert < 1 \\ \text{diverges for other } r \end{cases}$
p-Series	$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges for } p > 1 \\ \text{diverges for other } p \end{cases}$
Harmonic Series	$\sum_{k=1}^{\infty} \frac{1}{k}$ always diverges
Alternating Harmonic Series	$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges conditionally

Limits
$\lim_{k\to\infty} k^{1/k} = 1$
$\lim_{k\to\infty} \frac{x^k}{k!} = 0$
$\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e$
$\lim_{k\to\infty} (1+k)^{1/k} = 1$

Divergence, Convergence, Absolute Convergence and Conditional Convergence

- A series $\sum a_k$ is said to **converge absolutely** if $\sum \lvert a_k \rvert$ converges.
- If $\sum a_k$ converges but $\sum \lvert a_k \rvert$ diverges then $\sum a_k$ is said to **converge conditionally**.
- Every convergent series $\sum a_k$ is either absolutely or conditionally convergent, but we say it simply converges if we have not established that $\sum \lvert a_k \rvert$ converges or diverges.
- If a series does not converge it diverges.

Properties of Absolute and Conditional Convergence

- If a series converges *absolutely* to the value *L*, then any rearrangement of the series will also converge to the value *L*.
- If a series converges *conditionally* to the value *L*, then there exists a rearrangement of the series that converges to any other given real number, as well as rearrangements that diverge.