Polynomials and Factoring

Definitions

A **zero** is any value of *x*, real or complex, for which *function* $P(x)$ is zero. A **root** is any solution, real or complex, to the *equation* $P(x)=0$.

A **polynomial** function is a sum of monomials. Each monomial is the product of a constant and one or more variables, the variables being raised to a non-negative integer power. Since there is no restriction on the constants, they may be negative effectively making a sum a difference. The order or degree of a monomial is the sum of its variables' exponents. The polynomial's order is the same as its highest order monomial.

The remainder of this sheet only concerns itself with polynomial functions of one variable. These have the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$ where *n* is a non-negative integer and *a* is a constant.

Important Theorems

Polynomial Remainder Theorem: If polynomial $P(x)$ is divided by $x-b$ the remainder is $P(b)$.

Polynomial Factor Theorem: $x-b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$.

Fundamental Theorem of Algebra: A non-constant polynomial with complex coefficients has a least one complex zero. This can be extended using induction to show that a polynomial of degree $n \ge 1$ has exactly *n* complex zeros, counting repeated zeros. Keep in mind that complex numbers include real numbers.

Rational Root Theorem or Test: Given an n^{th} degree polynomial, $P(x) = sx^n + tx^{n-1} + \cdots + ux + v$ with integer only coefficents, if the rational number a/b in lowest terms is a zero of $P(x)$, then *a* will be a factor of of *v* and *b* will be a factor of *s*. We must include the negative factors of *s* and *v* as well.

The rational root theorem allows the factors of *s* and *v* to be tested to find potential rational zeros of the polynomial. Synthetic division an easy way to test a candidate zero and give the resulting quotient. The remainder theorem can also be used but does not give the quotient.

Synthetic division is a shorthand form of long division for the case where the divisor is in the form $(x-c)$. Since the divisor's *x* coefficient is 1, each quotient term's coefficient will always be the same as the dividend's. Since the constant term of the divisor is subtracted from *x* and the divisor is subtracted from the dividend, we can effectively add *c* to the dividend instead.

Find the factors of $2x^4 - 9x^3 + 7x^2 + 9x - 9$. Candidate rational zeros are: $\pm\frac{9,\ 3,\ 1}{1,\ 2}$. Let us try $x-3$ which means $x=3$:

$$\begin{array}{r|rrrrr} 3 & 2 & -9 & 7 & 9 & -9 \\ & & 6 & -9 & -6 & 9 \\ \hline & 2 & -3 & -2 & 3 & 0 \end{array}$$

This has zero remainder so $x-3$ is a factor. We can now work with the resulting cubic in the bottom row. We will try $-1/2$:

$$\begin{array}{r|rrrr} -1/2 & 2 & -3 & -2 & 3 \\ & & -1 & 2 & 0 \\ \hline & 2 & -4 & 0 & 3 \end{array}$$

This has a remainder so $x+1/2$ is not a factor. Let us try 1:

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -2 & 3 \\ & & 2 & -1 & -3 \\ \hline & 2 & -1 & -3 & 0 \end{array}$$

This has zero remainder so $x-1$ is a factor. This leaves the quadratic $2x^2 - 1x - 3$ which readily factors to $(2x-3)(x+1)$ and we conclude

$$2x^4 - 9x^3 + 7x^2 + 9x - 9 = (x-3)(x-1)(2x-3)(x+1).$$

Constraints on Possible Rational Roots

The following two sections count the number of sign reversals in the polynomial with the following convention for ordering the terms of the polynomial:

The polynomial is written from highest order to lowest order and only terms with nonzero coefficients are included. If we form a quotient by dividing a polynomial by another (lower order) polynomial, we also write the result in this manner, followed by its remainder, if any. We also only consider polynomials where the constant a_0 term is nonzero. This can be done by factoring out powers of x and then dealing with the resulting polynomial and x factor(s) separately. This removes any zeros at 0.

Descartes' Rule of Signs

Parity is a property of an integer. It is either even or odd. An integer has even parity if it is divisible by 2 with remainder 0; it has odd parity if the remainder is 1. For example, 1, 3, 5 have odd parity; 0, 2, 4 have even parity.

Zeros of Polynomial $P(x)$ Have These Two Properties
The number of **positive zero**s is $\le$ number of sign reversal of $P(x)$ AND has the same **parity** as the number of reversals
The number of **negative zeros** is $\le$ number of sign reversals of $P(-x)$ AND has the same **parity** as the number of reversals.

The rule of signs gives more information on what zeros not to consider when using the rational root test. Applying it to the last example we can create a table with the number of possible positive and negative zeros. Since we have already removed zeros at 0 any remaining zeros must be complex. There are 3 sign changes for $P(x)$ and 1 for $P(-x)$. This means there is exactly one negative zero and either 3 or 1 positive zeros. This is summarized in the table.

+	−	***i***
3	1	0
1	1	2

This helps because once a negative zero is found we need only look for positive zeros.

Upper and Lower Bound of Zeros

For a Polynomial $P(x)$ and the Quotient $P(x)/(x-c)$ and its Remainder
If c is **positive** and the terms of the quotient and its remainder are **the same sign** then c is an **upper bound** for real zeros of $P(x)$.
If c is **negative** and the terms of the quotient and its remainder **alternate sign**, then c is a **lower bound** for real zeros of $P(x)$.

The bounds test is also useful in narrowing the choices when applying the rational root test. In the example, had we first looked for a zero at $-3/2$ we would have

$$\begin{array}{r|rrrrr} -3/2 & 2 & -9 & 7 & 9 & -9 \\ & & -3 & 18 & -75/2 & 171/4 \\ \hline & 2 & -12 & 25 & -57/2 & 135/4 \end{array}$$

In addition to finding that $x+3/2$ is not a factor of $P(x)$, we can see that the terms of the quotient and remainder alternate sign and the zero being checked is negative. This tells us $-3/2$ is a lower bound for the zeros of this polynomial and there is no point in checking $-9/2$ or -9.