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Linear Approximation and Newton's Method

Linear Approximation

Given $f(x_0)$ if we want to find $f(x_1)$ and we know $\Delta y = f(x_1) - f(x_0)$ we can say $f(x_1) = f(x_0) + \Delta y$. If we know x_1 and x_0 are "near" each other we can make the linear approximation that $\Delta y \approx f'(x)\Delta x$:

> Linear approximation of $f(x_1)$ given $f(x_0)$ where x_1 and x_0 are "near": $f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0)$.

Example: Approximate $\cos 57°$ and $\cos 63°$. For the product involving the angle measure we need to convert to radians.

$$\cos 57° \approx \frac{1}{2} + \left(57\frac{\pi}{180} - \frac{\pi}{3}\right)(-\sin 60°)$$
$$\approx \frac{1}{2} + \pi\left(\frac{-3}{180}\right)\frac{-\sqrt{3}}{2}$$

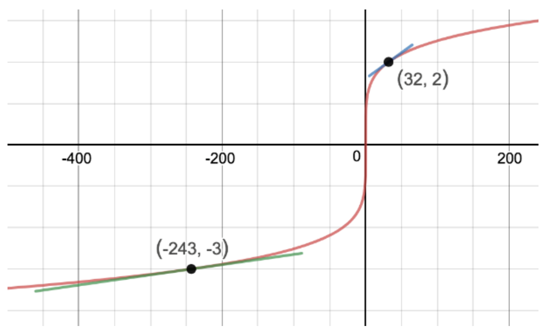
$\cos 57° \approx 0.54534$, an error of about 0.12%

$$\cos 63° \approx \frac{1}{2} + \left(63\frac{\pi}{180} - \frac{\pi}{3}\right)(-\sin 60°)$$
$$\approx \frac{1}{2} + \pi\left(\frac{3}{180}\right)\frac{-\sqrt{3}}{2}$$

$\cos 63° \approx 0.45466$, an error of about 0.15%

Example: What is the linear approximation of $\sqrt[5]{x}$ near 32; near −243. Graph these approximations.

$$\sqrt[5]{x} \approx \sqrt[5]{32} + \frac{1}{5 \cdot 32^{4/5}}(x - 32)$$
$$\approx \frac{1}{80}x + \frac{8}{5}$$
$$\sqrt[5]{x} \approx \sqrt[5]{-243} + \frac{1}{5 \cdot (-243)^{4/5}}(x - (-243))$$
$$\approx \frac{1}{405}x - \frac{12}{5}$$



The graph illustrates how the faster the curve is changing the smaller the region in which a given approximation is reasonable.

Newton's Method

If we want to find the zeros of f we can use the linear approximation formula $f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0)$: Start with an initial guess x_0, take $f(x_1) = 0$ then solve for x_1. Repeat the process to find successively better approximations of the root. The iterative formula is show below.

> Newton's Method for finding roots of f: $x_{n+1} \approx x_n - \dfrac{f(x_n)}{f'(x_n)}$

Assuming a root exists, there is no guarantee this method will find it, nor, if there are multiple roots, which root it will find. The method is very dependent on the value of the initial guess relative to the location of the roots and the first and second derivatives of the function. If the graph of the function can be inspected, it is possible to make a good initial guess: one near the root being sought and where $|f'|$ is not too small (so we are not dividing by a small number) and $|f''|$ is not too large (so that the curve is not so sharp that the linear approximation is poor).

Example: Find the solution to $2^x = 4x$. This is the same as asking for the roots of $f(x) = 2^x - 4x$. If you graph this function you see that there are two roots. The iterations below find each root by selecting different initial guesses.

n	x_{n+1}	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	x_n	$f(x_n)$	$f'(x_n)$
0	-2.17162	1.10000	-1.05163	-0.32144	2.24733	1.40000	-0.94446	1.11463
1	0.07991	-2.17162	8.77851	-3.89891	1.93304	2.24733	2.82069	8.97463
2	0.35561	0.07991	0.77213	-2.80058	1.81165	1.93304	0.62953	5.18625
3	0.37898	0.35561	0.05554	-2.37627	1.79404	1.81165	0.07115	4.03939
4	0.37919	0.37898	0.00049	-2.33404	1.79369	1.79404	0.00136	3.88531
5	0.37919	0.37919	0.00000	-2.33365	1.79369	1.79369	0.00000	3.88228

The function has a turning point near 1.1763 (the derivative is zero). If the initial guess was at or near this point the method would fail because of division by zero.