Limits, Derivatives and Integrals

Assumptions

All functions, variables and constants are real valued.

f is a function. v and u are functions of a common independent variable.

x is a real valued variable; a, b, c and n are real number constants.

Limits

$\lim_{x\to 0}\frac{\sin x}{x}=1$	$\lim_{x\to 0}\frac{1-\cos x}{x}=0$	$\lim_{x\to 0}\frac{e^x-1}{x}=1$	$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x=1$	$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\triangleq e$

Derivative Definition and Properties

Definition	General: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	At the point a: $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

Properties	$(cu)'=cu'$	$(u\pm v)'=u'\pm v'$	$(uv)'=u'v+uv'$	$\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$

Chain Rule	$\left[f(u(x))\right]'=f'(u(x))u'(x)$

L'Hôpital's Rule

Suppose f and g are differentiable functions on an open interval containing a, but not necessarily at a, and $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$ or $\pm\infty$ and in the interval $g'(x)\neq 0$ except possibly at a, if $\lim_{x\to a}\left[f'(x)/g'(x)\right]$ exists, or is $\pm\infty$ then $\lim_{x\to a}f(x)/g(x)=\lim_{x\to a}f'(x)/g'(x)$. Moreover, this holds for the limits: $x\to a^-$, $x\to a^+$ or $x\to\pm\infty$.

Derivative of Basic Functions

If x is replaced with a function in the derivatives below *the chain rule must be applied*.

Constant, Power, Exponential and Logarithmic Functions

$c'=0$	$(x^n)'=nx^{n-1}$	$\left(e^x\right)'=e^x$	$\left(b^x\right)'=(\ln b)b^x \quad b>0$	$\left(\ln\lvert x\rvert\right)'=\frac{1}{x},\ x\neq 0$	$\left(\log_b\lvert x\rvert\right)'=\frac{1}{x\ln b},\ \ x\neq 0,\ b>0$

Trigonometric Functions

$(\sin x)'=\cos x$	$(\cos x)'=-\sin x$
$(\tan x)'=\sec^2 x$	$(\cot x)'=-\csc^2 x$
$(\sec x)'=\sec x\tan x$	$(\csc x)'=-\csc x\cot x$

Inverse Trigonometric Functions

$\left(\sin^{-1}x\right)'=\frac{1}{\sqrt{1-x^2}}$	$\left(\cos^{-1}x\right)'=\frac{-1}{\sqrt{1-x^2}}$
$\left(\tan^{-1}x\right)'=\frac{1}{1+x^2}$	$\left(\cot^{-1}x\right)'=\frac{-1}{1+x^2}$
$\left(\sec^{-1}x\right)'=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\left(\csc^{-1}x\right)'=\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

Functions as Limits of Integration

Given functions f, g and h are integrable and function F is an antiderivative of f the Fundamental Theorem of Calculus tells us

$$\int_{h(x)}^{g(x)} f(t)dt = \int_0^{g(x)} f(t)dt - \int_0^{h(x)} f(t)dt = F\left(g(x)\right) - F(0) - F\left(h(x)\right) + F(0) = F\left(g(x)\right) - F\left(h(x)\right).$$

Applying the chain rule of differentiation gives,

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = \frac{d}{dx}\left[F\left(g(x)\right) - F\left(h(x)\right)\right] = \frac{d}{dx}\left[F\left(g(x)\right)\right] - \frac{d}{dx}\left[F\left(h(x)\right)\right] = f\left(g(x)\right)g'(x) - f\left(h(x)\right)h'(x).$$

Integrals of Basic Functions

Constant, Power, Exponential, 1/*x* and Logarithmic Functions

$\int c\,dx = cx + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ \ n \neq 1$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C,\ \ x \neq 0$	$\int e^x dx = e^x + C$
$\int b^x dx = \frac{b^x}{\ln b} + C\ \ b > 0$	$\int \ln x\,dx = x\ln x - x + C,\ \ x > 0$	$\int \log_b x\,dx = \frac{1}{\ln b}\left(x\ln x - x\right) + C,\ \ x > 0,\ \ b > 0$	

Trigonometric Functions

$\int \sin x\,dx = -\cos x + C$	$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert + C = \ln\lvert\tan\left(x/2\right)\rvert + C$
$\int \cos x\,dx = \sin x + C$	$\int \cot dx = \ln\lvert\sin x\rvert + C$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C = \ln\lvert\tan\left(\pi/4 + x/2\right)\rvert + C$

Inverse Trigonometric Functions

$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$	$\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$
$\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{x^2+1} + C$	$\int \cot^{-1} x\,dx = x\cot^{-1} x + \ln\sqrt{x^2+1} + C$
$\int \sec^{-1} x\,dx = x\sec^{-1} x - \ln\left\lvert x + \sqrt{x^2-1}\right\rvert + C$	$\int \csc^{-1} x\,dx = x\csc^{-1} x + \ln\left\lvert x + \sqrt{x^2-1}\right\rvert + C$

Some Functions containing $x^2 \pm a^2$ and $a^2 - x^2$

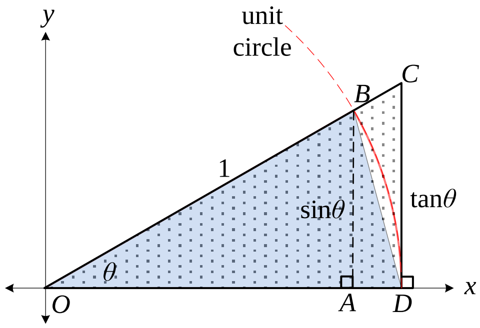
$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left\lvert x + \sqrt{x^2-a^2}\right\rvert + C$	$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right) + C$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C = -\cos^{-1}\frac{x}{a} + C$
$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$	$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C$	
$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left\lvert x + \sqrt{x^2-a^2}\right\rvert + C$		$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

Trigonometric Reductions

$\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$	$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$
$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$	$\int \cot^n x\,dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx$
$\int \csc^n x\,dx = \frac{-1}{n-1}\csc^{n-2} x\cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx$	$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

Proof of Limits

Proof $\lim_{x\to 0}\frac{\sin x}{x}$.



In the diagram to the left quadrant I of the unit circle is shown with two triangles and a sector drawn on the same angle θ. Since it is a unit circle the lengths $OB = OD = 1$. The blue triangle $\triangle OBD$ has area $(\sin\theta)/2$. The dotted triangle $\triangle OCD$ has area $(\tan x)/2$. The sector formed by the arc BD has area $\pi\theta/(2\pi) = \theta/2$

The area of $\triangle OBD$ is a subset of the area of the sector which in turn is a subset of the area of $\triangle OCD$. This allows us to write,

$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2}$.

We need to consider a negative θ in quadrant IV so we can take the two-sided limit as $\theta \to 0$. Since area is taken to be always positive, we add absolute value signs to each area to ensure the area remains positive in quadrant IV,

$$\left|\frac{\sin\theta}{2}\right| \le \left|\frac{\theta}{2}\right| \le \left|\frac{\tan\theta}{2}\right|$$

Multiplying this equation by $|2/\sin\theta|$ gives,

$$1 \le \left|\frac{\theta}{\sin\theta}\right| \le \left|\frac{1}{\cos\theta}\right|.$$

In quadrant I and IV $\cos\theta$ is positive and both θ and $\sin\theta$ have the same sign. This means all terms in the inequality are positive and the absolute value signs can be removed,

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}.$$

Taking the reciprocal, we reverse the inequality signs.

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

Apply the squeeze theorem with the limit $x \to 0$.

$$\lim_{x\to 0} 1 \ge \lim_{x\to 0}\frac{\sin x}{x} \ge \lim_{x\to 0}\cos x$$

$$1 \ge \lim_{x\to 0}\frac{\sin x}{x} \ge 1$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

□

Proof $\lim_{x\to 0}\frac{1-\cos x}{x}=0$.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\frac{1+\cos x}{1+\cos x}\\ &= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\frac{\sin x}{(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{(1+\cos x)}\\ &= 1\cdot 0\\ &= 0\end{aligned}$$

□

Proof $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

Use the definition

$$e \triangleq \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \;\to\; e=\lim_{x\to 0}(1+x)^{1/x}$$

to write,

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1}{x} &= \lim_{x\to 0}\frac{\left(\lim_{x\to 0}(1+x)^{1/x}\right)^x-1}{x}\\ &= \lim_{x\to 0}\frac{\left((1+x)^{1/x}\right)^x-1}{x}\\ &= \lim_{x\to 0}\frac{(1+x)^{x/x}-1}{x}\\ &= \lim_{x\to 0}\frac{1+x-1}{x}\\ &= \lim_{x\to 0}\frac{x}{x}\\ &= \lim_{x\to 0}1\\ &= 1\end{aligned}$$

□

Proof $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x=1$.

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = e^{\ln\left(\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x\right)}$$

Focusing on the exponent on the right above and utilizing L'Hôpital's rule we have,

$$\begin{aligned}\ln\left(\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x\right) &= \lim_{x\to 0}\ln\left(1+\frac{1}{x}\right)^x\\ &= \lim_{x\to 0}x\ln\left(1+\frac{1}{x}\right)\\ &= \lim_{x\to 0}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\\ &= \lim_{x\to 0}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}\\ &= \lim_{x\to 0}\frac{\left(\frac{1}{1+1/x}\right)\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}}\\ &= \lim_{x\to 0}\frac{1}{1+1/x}\\ &= 0.\end{aligned}$$

Replace this result for the exponent in the original equation,

$$\begin{aligned}\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x &= e^0\\ &= 1\end{aligned}$$

□

Proof of Derivatives

Derivatives of Power Function

For $n \in \mathbb{Z}$ and $n > 0$:

$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$, use binomial expansion for $(x+h)^n$ to write

$$
\begin{aligned}
&= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h} \\
&= \lim_{h\to 0}\left(\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right) \\
&= \binom{n}{1}x^{n-1} \\
&= nx^{n-1}
\end{aligned}
$$

For $n \in \mathbb{Z}$ and $n < 0$ let $m = -n$ so $m > 0$ and the previous result can be used along with the quotient rule:

$$x^n = x^{-m} = \frac{1}{x^m}$$

$$
\begin{aligned}
\frac{d}{dx}x^n &= \frac{0\cdot x^m - mx^{m-1}\cdot 1}{x^{2m}} \\
&= -mx^{m-1-2m} \\
&= -mx^{-m-1} \\
&= nx^{n-1}
\end{aligned}
$$

□

$\frac{d}{dx}\left(x^n\right)$, where $n \in \mathbb{R}$ and $n \neq 0$. Use implicit differentiation after taking the logarithm.

$x > 0$ or $x < 0$ and n even so $x^n = |x|^n > 0$.

$$
\begin{aligned}
y &= x^n = |x|^n \\
\ln y &= \ln|x|^n \\
\ln y &= n\ln|x| \\
\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{x} \\
\frac{dy}{dx} &= \frac{n\cdot y}{x} \\
&= \frac{n\cdot x^n}{x} \\
&= nx^{n-1}
\end{aligned}
$$

□

$x < 0$ and n odd so $x^n = -|x|^n < 0$.

$$
\begin{aligned}
y &= x^n = -|x|^n \\
-y &= |x|^n \\
\ln(-y) &= \ln|x|^n \\
\ln(-y) &= n\ln|x| \\
\frac{1}{-y}\left(-\frac{dy}{dx}\right) &= n\frac{1}{x} \\
\frac{dy}{dx} &= \frac{n\cdot y}{x} \\
\frac{dy}{dx} &= \frac{n\cdot x^n}{x} \\
&= nx^{n-1}
\end{aligned}
$$

□

Derivatives of Logarithmic and Exponential Functions

$$\frac{d}{dx}\left(e^x\right)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$
$$=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}$$
$$=e^x\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}$$

Euler's number e can be defined by the limit: $e\triangleq\lim_{n\to\infty}\left(1+1/n\right)^n$.

By substituting $h=1/n$ this can be rewritten as $e=\lim_{h\to 0}\left(1+h\right)^{1/h}$.

Substituting this for e above in the right side gives,

$$\frac{d}{dx}\left(e^x\right)=e^x\lim_{h\to 0}\frac{\left(\left(\lim_{h\to 0}\left(1+h\right)^{1/h}\right)^h-1\right)}{h}$$
$$=e^x\lim_{h\to 0}\frac{\left(\left(1+h\right)^{h/h}-1\right)}{h}$$
$$=e^x\lim_{h\to 0}\frac{\left(1+h-1\right)}{h}$$
$$=e^x\lim_{h\to 0}\frac{h}{h}$$
$$=e^x\lim_{h\to 0}1$$
$$=e^x$$

□

Use implicit differentiation:

$$y=\ln x$$
$$e^y=x$$
$$e^y\frac{dy}{dx}=1$$
$$\frac{dy}{dx}=\frac{1}{e^y}$$
$$\frac{d}{dx}\left(\ln x\right)=\frac{1}{x}$$

□

$$\frac{d}{dx}\left(b^x\right)=\frac{d}{dx}\left(e^{\ln b^x}\right)$$
$$=\frac{d}{dx}\left(e^{x\ln b}\right)$$
$$=e^{x\ln b}\ln b$$
$$=e^{\ln b^x}\ln b$$
$$=b^x\ln b$$

□

$$\frac{d}{dx}\left(\log_b x\right)=\frac{d}{dx}\left(\frac{\ln x}{\ln b}\right)$$
$$=\frac{1}{\ln b}\frac{d}{dx}\left(\ln x\right)$$
$$=\frac{1}{x\ln b}$$

□

Derivatives of Trigonometric Functions

$\frac{d}{dx}(\sin x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$ $=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$ $=\lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}$ $=\sin x\cdot\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}$ $=\sin x\cdot 0+(\cos x)1$ $=\cos x$ □	$\frac{d}{dx}(\cos x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$ $=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$ $=\lim_{h\to 0}\frac{\cos x\cos h-\cos x}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h}$ $=\cos x\cdot\lim_{h\to 0}\frac{(\cos h-1)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$ $=\cos x\cdot 0-(\sin x)1$ $=-\sin x$ □
$\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ $=\frac{\cos x\cdot\frac{d}{dx}\sin x-\sin x\frac{d}{dx}\cos x}{\cos^2 h}$ $=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$ $=\frac{1}{\cos^2 x}$ $=\sec^2 x$ □	$\frac{d}{dx}(\cot x)=\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ $=\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}$ $=\frac{-\left(\sin^2 x+\cos^2 x\right)}{\sin^2 x}$ $-\frac{1}{\sin^2 x}$ $=-\csc^2 x$ □
$\frac{d}{dx}(\sec x)=\frac{d}{dx}\left(\frac{1}{\cos x}\right)$ $=\frac{0-(-\sin x)}{\cos^2 x}$ $=\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$ $=\sec x\tan x$ □	$\frac{d}{dx}(\csc x)=\frac{d}{dx}\left(\frac{1}{\sin x}\right)$ $=\frac{0-\cos x}{\sin^2 x}$ $=-\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}$ $=-\csc x\cot x$ □

Derivatives of Inverse Trigonometric Functions

$y=\sin^{-1}x \;\to\; \cos^2 y = 1-x^2$ $\sin y = x$ $\cos y \dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$ $\cos y$ is always postive on the range of $\sin^{-1}x$, $y\in[-\pi/2,\pi/2]$. This allows us to place the denominator in absolute value signs. $\dfrac{dy}{dx} = \dfrac{1}{\lvert\cos y\rvert}$ $= \dfrac{1}{\sqrt{1-x^2}}$ □	$y=\cos^{-1}x \;\to\; \sin^2 y = 1-x^2$ $\cos y = x$ $-\sin y \dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{-1}{\sin y}$ $\sin y$ is always postive on the range of $\cos^{-1}x$, $y\in[0,\pi]$. This allows us to place the denominator in absolute value signs. $\dfrac{dy}{dx} = \dfrac{-1}{\lvert\sin y\rvert}$ $= \dfrac{-1}{\sqrt{1-x^2}}$ □	$y=\tan^{-1}x \;\to\; \sec^2 y = x^2+1$ $\tan y = x$ $\sec^2 y \dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$ $= \dfrac{1}{1+x^2}$ □
$y=\csc^{-1}x \;\to\; \cot^2 y = x^2-1$ $\csc y = x$ $-\csc y \cot y \dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{-1}{\csc y \cot y}$ The domain of $x=\csc y$ is $y\in[-\pi/2,0)\cup(0,\pi/2]$. On this domain $\csc y$ and $\cot y$ are always the same sign so their product is always postive. This allows us to place the denominator in absolute value signs. $\dfrac{dy}{dx} = \dfrac{-1}{\lvert\csc y \cot y\rvert}$ $= \dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$ □	$y=\sec^{-1}x \;\to\; \tan^2 y = x^2-1$ $\sec y = x$ $\sec y \tan y \dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$ The domain of $x=\sec y$ is $y\in[0,\pi/2)\cup(\pi/2,\pi]$. On this domain $\sec y$ and $\tan y$ are always the same sign so their product is always postive. This allows us to place the denominator in absolute value signs. $\dfrac{dy}{dx} = \dfrac{1}{\lvert\sec y \tan y\rvert}$ $= \dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$ □	$y=\cot^{-1}x \;\to\; \csc^2 y = 1+x^2$ $\cot y = x$ $-\csc^2 y \dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{-1}{\csc^2 y}$ $= \dfrac{-1}{1+x^2}$ □

Proof of Integrals

Integral of Trigonometric Functions

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= \int \frac{\sin x}{u} \frac{du}{-\sin x}; \quad u = \cos x \quad \to \quad du = -\sin x dx \\
&= -\int \frac{1}{u} du \\
&= -\ln|u| \\
&= -\ln|\cos x| + C \\
&= \ln|\sec x| + C
\end{aligned}$$

□

$$\begin{aligned}
\int \cot x dx &= \int \frac{\cos x}{\sin x} dx \\
&= \int \frac{\cos x}{u} \frac{du}{\cos x}; \quad u = \sin x \quad \to \quad du = \cos x dx \\
&= \int \frac{1}{u} du \\
&= \ln|u| \\
&= \ln|\sin x| + C
\end{aligned}$$

□

$$\begin{aligned}
\int \csc x dx &= \int \frac{1}{\sin x} dx \\
&= \int \frac{\sin x}{\sin^2 x} dx \\
&= \int \frac{\sin x}{1-\cos^2 x} dx \\
&= \int \frac{\sin x}{1-u^2} \frac{-du}{\sin x}; \quad u = \cos x \quad \to \quad du = -\sin x dx \\
&= \int \frac{1}{u^2-1} du \\
&= \int \left(\frac{-1/2}{u+1} - \frac{-1/2}{u-1} \right) du; \quad \text{using partial fraction expansion} \\
&= -\frac{1}{2} \left(\int \frac{du}{u+1} - \int \frac{du}{u-1} \right) \\
&= -\frac{1}{2} \left(\ln|u+1| - \ln|u-1| \right) \\
&= \frac{1}{2} \ln\left| \frac{u-1}{u+1} \right| \\
&= \frac{1}{2} \ln\left| \frac{u-1}{u+1} \frac{u-1}{u-1} \right| \\
&= \frac{1}{2} \ln\left| \frac{(u-1)^2}{u^2-1} \right| \\
&= \frac{1}{2} \ln\left| \frac{(\cos x-1)^2}{\cos^2 x-1} \right| \\
&= \frac{1}{2} \ln\left| \frac{(\cos x-1)^2}{-\sin^2 x} \right| \\
&= \ln\left(\frac{(\cos x-1)^2}{\sin^2 x} \right)^{1/2}; \quad \text{using } \left|-\sin^2 x\right| = \sin^2 x \\
&= \ln\left| \frac{\cos x-1}{\sin x} \right|; \quad \text{use } |\ | \text{ to maintain postive root} \\
&= \ln|\cot x - \csc x| + C = \ln|\csc x - \cot x| + C
\end{aligned}$$

□

$$\begin{aligned}
\int \sec x dx &= \int \frac{1}{\cos x} dx \\
&= \int \frac{\cos x}{\cos^2 x} dx \\
&= \int \frac{\cos x}{1-\sin^2 x} dx \\
&= \int \frac{\cos x}{1-u^2} \frac{du}{\cos x}; \quad u = \sin x \quad \to \quad du = \cos x dx \\
&= \int \frac{1}{1-u^2} du \\
&= \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u} \right) du; \quad \text{using partial fraction expansion} \\
&= \frac{1}{2} \left(\int \frac{du}{1+u} + \int \frac{du}{1-u} \right) \\
&= \frac{1}{2} \left(\ln|1+u| - \ln|1-u| \right) \\
&= \frac{1}{2} \ln\left| \frac{1+u}{1-u} \right| \\
&= \frac{1}{2} \ln\left| \frac{1+u}{1-u} \frac{1+u}{1+u} \right| \\
&= \frac{1}{2} \ln\left| \frac{(1+u)^2}{1-u^2} \right| \\
&= \frac{1}{2} \ln\left| \frac{(1+\sin x)^2}{1-\sin^2 x} \right| \\
&= \frac{1}{2} \ln\left| \frac{(1+\sin x)^2}{\cos^2 x} \right| \\
&= \ln\left(\frac{(1+\sin x)^2}{\cos^2 x} \right)^{1/2} \\
&= \ln\left| \frac{1+\sin x}{\cos x} \right|; \quad \text{use } |\ | \text{ to maintain postive root} \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

□

Integral of Inverse Trigonometric Functions

$\int \sin^{-1} x dx = \int u dv = uv - \int v du$

where $u = \sin^{-1} x$ and $dv = dx$

so that $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$

$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

let $w = 1 - x^2 \rightarrow dw = -2x dx$

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{-2x\sqrt{w}} dw;$$
$$= x \sin^{-1} x + \frac{1}{2} \int \frac{1}{\sqrt{w}} dw$$
$$= x \sin^{-1} x + \frac{1}{2} 2\sqrt{w}$$
$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

□

$\int \csc^{-1} x dx = \int u dv = uv - \int v du; \quad u = \csc^{-1} x, \quad dv = dx \quad \rightarrow \quad du = \frac{-1}{x\sqrt{x^2-1}} dx, \ v = x$

$$= x \csc^{-1} x + \int \frac{x}{x\sqrt{x^2-1}} dx$$
$$= x \csc^{-1} x + \int \frac{1}{\sqrt{x^2-1}} dx$$

Let $x = \csc u$ so $dx = -\csc u \cot u \, du$ and $\cot u = \sqrt{x^2-1}$

$$= x \csc^{-1} x + \int \frac{-\csc u \cot u}{\sqrt{\csc u^2 - 1}} du$$
$$= x \csc^{-1} x - \int \frac{\csc u \cot u}{\cot u} du$$
$$= x \csc^{-1} x - \int \csc u \, du$$
$$= x \csc^{-1} x + \ln\left|\cot u + \csc u\right|$$
$$= x \csc^{-1} x + \ln\left|x + \sqrt{x^2-1}\right|$$

□

$\int \cos^{-1} x dx = \int u dv = uv - \int v du$

where $u = \cos^{-1} x$ and $dv = dx$

so that $du = \frac{-1}{\sqrt{1-x^2}} dx$ and $v = x$

$\int \cos^{-1} x dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx$

see $\int \sin^{-1} x dx$ for solution to $\int \frac{x}{\sqrt{1-x^2}} dx$

$\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$

□

$\int \sec^{-1} x dx = \int u dv = uv - \int v du; \quad u = \sec^{-1} x, \quad dv = dx \quad \rightarrow \quad du = \frac{1}{x\sqrt{x^2-1}} dx, \ v = x$

$$= x \sec^{-1} x - \int \frac{x}{x\sqrt{x^2-1}} dx$$
$$= x \sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}} dx$$
$$= x \sec^{-1} x - \int \frac{\sec u \tan u}{\sqrt{\sec u^2 - 1}} du; \quad x = \sec u \rightarrow \quad dx = \sec u \tan u du$$
$$= x \sec^{-1} x - \int \frac{\sec u \tan u}{\tan u} du$$
$$= x \sec^{-1} x - \int \sec u du$$
$$= x \sec^{-1} x - \ln\left|\sec u + \tan u\right|; \quad x = \sec u \quad \rightarrow \quad \tan u = \sqrt{x^2-1}$$
$$= x \sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right|$$

□

$\int \tan^{-1} x dx = \int u dv = uv - \int v du;$

where $u = \tan^{-1} x$ and $dv = dx$

so that $du = \frac{1}{1+x^2} dx$ and $v = x$

$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

Let $w = 1 + x^2 \rightarrow dw = 2x dx$

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{2xw} dw$$
$$= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{w} dw$$
$$= x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right)$$
$$= x \tan^{-1} x - \ln\sqrt{1+x^2} + C$$

□

$\int \cot^{-1} x dx = \int u dv = uv - \int v du; \quad u = \cot^{-1} x, \quad dv = dx \quad \rightarrow \quad du = \frac{-1}{1+x^2} dx, \ v = x$

$= x \cot^{-1} x + \int \frac{x}{1+x^2} dx$, see $\int \tan^{-1} x dx$ for details on solving this integral

$$= x \cot^{-1} x + \ln\sqrt{1+x^2} + C$$

□

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx \quad \textbf{and} \quad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

Let $x = a\csc u \rightarrow dx = -a\csc u\cot u\,du$

and $\cot u = \sqrt{x^2-a^2}/a$, $u \in [-\pi/2, 0) \cup (0, \pi/2]$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\,dx &= \int \frac{1}{a\sqrt{\csc^2 u - 1}} - a\csc u\cot u\,du \\
&= -\int \frac{1}{\cot u}\csc u\cot u\,du \\
&= -\int \csc u\,du \\
&= -\ln\left|\csc u - \cot u\right| + C_0 \\
&= -\ln\left|\frac{x}{a} - \frac{\sqrt{x^2-a^2}}{a}\right| + C_0 \\
&= -\ln\left|\frac{x-\sqrt{x^2-a^2}}{a}\,\frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}\right| + C_0 \\
&= -\ln\left|\frac{a}{x+\sqrt{x^2-a^2}}\right| + C_0 \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| - \ln a + C_0 \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| + C
\end{aligned}$$

□

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

Let $x = a\tan u \rightarrow dx = a\sec^2 u\,du$ and $\sec u = \sqrt{x^2+a^2}/a$,

$u \in [0, \pi/2) \cup (0, \pi]$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \frac{1}{a\sqrt{\tan^2 u + 1}} a\sec^2 u\,du \\
&= \int \frac{1}{\sec u}\sec^2 u\,du \\
&= \int \sec u\,du \\
&= \ln\left|\sec u + \tan u\right| + C_0 \\
&= \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C_0 \\
&= \ln\left|\sqrt{x^2+a^2}+x\right| - \ln a + C_0 \\
&= \ln\left|\sqrt{x^2+a^2}+x\right| + C \\
&= \ln\left(\sqrt{x^2+a^2}+x\right) + C
\end{aligned}$$

□

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

Let $u = x/a \rightarrow x = au$ and $dx = a\,du$

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{a}{\sqrt{a^2-a^2u^2}}\,du \\
&= \int \frac{1}{\sqrt{1-u^2}}\,du \\
&= \sin^{-1} u + C \\
&= \sin^{-1} x/a + C
\end{aligned}$$

Since $\sin^{-1}\theta = \pi/2 - \cos^{-1}\theta$,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1} x/a + C = -\cos^{-1} x/a + C$$

□

$\int \frac{1}{x^2 \pm a^2}dx$, $\int \sqrt{x^2-a^2}\,dx$ and $\int \sqrt{a^2-x^2}\,dx$

$\int \frac{1}{x^2-a^2}dx$

Use the partial fraction expansion $\frac{1}{x^2-a^2} = \frac{1/(2a)}{x-a} + \frac{-1/(2a)}{x+a}$.

$$\int \frac{1}{x^2-1}dx = \frac{1}{2a}\left(\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right)$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

□

$\int \frac{1}{x^2+a^2}dx$

Let $u = x/a \;\rightarrow\; x = au$ and $dx = adu$

$$\int \frac{1}{x^2+a^2}dx = \int \frac{a}{a^2u^2+a^2}du$$

$$= \frac{1}{a}\int \frac{1}{u^2+1}du$$

$$= \frac{1}{a}\tan^{-1}u + C$$

$$= \frac{1}{a}\tan^{-1}x/a + C$$

Since $\tan^{-1}\theta = \pi/2 - \cot^{-1}\theta$,

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\tan^{-1}x/a + C = -\frac{1}{a}\cot^{-1}x/a + C$$

□

$\int \sqrt{x^2-a^2}\,dx$

Let $x = a\sec u \;\rightarrow\; dx = a\sec u\tan u\,du$ and $\tan u = \frac{\sqrt{x^2-a^2}}{a}$.

$$\int \sqrt{x^2-a^2}\,dx = \int \sqrt{a^2\sec^2 u - a^2}\,(a\sec u\tan u)\,du$$

$$= \int (a\tan u)a\sec u\tan u\,du$$

$$= a^2\int \sec u\tan^2 u\,du$$

Rewrite the integrand using the Pythagorean identity,

$$\sec u\tan^2 u = \frac{\sin^2 u}{\cos^3 u} = \frac{1-\cos^2 u}{\cos^3 u} = \sec^3 u - \sec u$$

$$\int \sqrt{x^2-a^2}\,dx = a^2\left(\int \sec^3 u\,du - \int \sec u\,du\right)$$

$$= a^2\left(\frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u+\tan u| - \ln|\sec u+\tan u|\right) + C_0$$

$$= \frac{a^2}{2}\sec u\tan u - \frac{1}{2}a^2\ln|\sec u+\tan u| + C_0$$

$$= \frac{a^2}{2}\frac{x}{a}\frac{\sqrt{x^2-a^2}}{a} - \frac{1}{2}a^2\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_0$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{3}{2}a^2\ln|a| + C_0$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

□

$\int \sqrt{a^2-x^2}\,dx$

Let $x = a\sin u \;\rightarrow\; dx = a\cos u\,du$. This means

$u = \sin^{-1}\frac{x}{a}$,

$\sin u = \frac{x}{a}$ and $\cos u = \frac{\sqrt{a^2-x^2}}{a}$

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2 u}\,(a\cos u)\,du$$

$$= \int \sqrt{a^2\cos^2 u}\,(a\cos u)\,du$$

$$= a^2\int \cos^2 u\,du$$

Use identity $\cos^2 u = \frac{1+\cos 2u}{2}$,

$$= \frac{a^2}{2}\int du + \frac{a^2}{2}\int \cos 2u\,du$$

$$= \frac{a^2}{2}u + \frac{a^2}{4}\sin 2u$$

Use identity $\sin 2u = 2\sin u\cos u$,

$$= \frac{a^2}{2}u + \frac{a^2}{4}2\sin u\cos u$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2-x^2}}{a}\right)$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

□

Integral Reduction of Powers of Trigonometric Functions

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$$

Let $u = \sin^{n-1} x$ and $dv = \sin x dx \rightarrow du = (n-1)\sin^{n-2} x \cos x dx$ and $v = -\cos x$

$$\begin{aligned} \int \sin^n x dx &= \int u dv \\ &= uv - \int v du \\ &= -\cos x \sin^{n-1} x + \int \cos x (n-1)\sin^{n-2} x \cos x dx \\ &= -\cos x \sin^{n-1} x + \int \cos^2 x (n-1)\sin^{n-2} x dx \\ &= -\cos x \sin^{n-1} x + \int (1-\sin^2 x)(n-1)\sin^{n-2} x dx \\ &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx \end{aligned}$$

$$n\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

$$\int \sin^n x dx = \frac{-1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x dx \quad \square$$

$$\int \cos^n x dx = \int \cos^{n-1} x \cos x dx$$

Let $u = \cos^{n-1} x$ and $dv = \cos x dx \rightarrow$ $du = -(n-1)\cos^{n-2} x \sin x dx$ and $v = \sin x$

$$\begin{aligned} \int \cos^n x dx &= \int u dv \\ &= uv - \int v du \\ &= \cos^{n-1} x \sin x + (n-1)\int \sin x \cos^{n-2} x \sin x dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x dx \\ &= \cos^{n-1} x \sin x + (n-1)\cos^{n-2} x dx - (n-1)\int \cos^n x dx \end{aligned}$$

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx \quad \square$$

$$\begin{aligned} \int \tan^n x dx &= \int \tan^{n-2} x \tan^2 x dx \\ &= \int \tan^{n-2} x (\sec^2 x - 1) dx \\ &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \end{aligned}$$

Let $u = \tan^{n-2} x$ and $dv = \sec^2 x dx \rightarrow$ $du = (n-2)\tan^{n-3} x \sec^2 x dx$ and $v = \tan x$

$$\begin{aligned} \int \tan^{n-2} x \sec^2 x dx &= \int u dv \\ &= uv - \int v du \\ &= \tan^{n-1} x - \int \tan x (n-2)\tan^{n-3} x \sec^2 x dx \\ &= \tan^{n-1} x - (n-2)\int \tan^{n-2} x \sec^2 x dx \end{aligned}$$

$$(n-1)\int \tan^{n-2} x \sec^2 x dx = \tan^{n-1} x$$

$$\int \tan^{n-2} x \sec^2 x dx = \frac{1}{n-1}\tan^{n-1} x$$

$$\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx \quad \square$$

$$\begin{aligned} \int \cot^n x dx &= \int \cot^{n-2} x \cot^2 x dx \\ &= \int \cot^{n-2} x (\csc^2 x - 1) dx \\ &= \int \cot^{n-2} x \csc^2 x dx - \int \cot^{n-2} x dx \end{aligned}$$

Let $u = \cot^{n-2} x$ and $dv = \csc^2 x dx \rightarrow$ $du = -(n-2)\cot^{n-3} x \csc^2 x dx$ and $v = -\cot x$

$$\begin{aligned} \int \cot^{n-2} x \csc^2 x dx &= \int u dv \\ &= uv - \int v du \\ &= -\cot^{n-1} x - \int \cot x (n-2)\cot^{n-3} x \csc^2 x dx \\ &= -\cot^{n-1} x - (n-2)\int \cot^{n-2} x \csc^2 x dx \end{aligned}$$

$$(n-1)\int \cot^{n-2} x \csc^2 x dx = -\cot^{n-1} x$$

$$\int \cot^{n-2} x \csc^2 x dx = \frac{-1}{n-1}\cot^{n-1} x$$

$$\int \cot^n x dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x dx \quad \square$$

$$\int \csc^n x dx = \int \csc^{n-2} x \csc^2 x dx$$

Let $u = \csc^{n-2} x$ and $dv = \csc^2 x dx \rightarrow$ $du = -(n-2)\csc^{n-2} x \cot x dx$ and $v = -\cot x$

$$\begin{aligned} \int \csc^n x dx &= \int u dv \\ &= uv - \int v du \\ &= -\csc^{n-2} x \cot x - (n-2)\int \cot^2 x \csc^{n-2} x dx \\ &= -\csc^{n-2} x \cot x - (n-2)\int (\csc^2 - 1)\csc^{n-2} x dx \\ &= -\csc^{n-2} x \cot x - (n-2)\left(\int \csc^n x dx - \int \csc^{n-2} x dx\right) \\ &= -\csc^{n-2} x \cot x - (n-2)\int \csc^n x dx + (n-2)\int \csc^{n-2} x dx \end{aligned}$$

$$(n-1)\int \csc^n x dx = -\csc^{n-2} x \cot x + (n-2)\int \csc^{n-2} x dx$$

$$\int \csc^n x dx = \frac{-1}{n-1}\csc^{n-2} x \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x dx \quad \square$$

$$\int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$$

Let $u = \sec^{n-2} x$ and $dv = \sec^2 x dx \rightarrow$ $du = (n-2)\sec^{n-2} x \tan x dx$ and $v = \tan x$

$$\begin{aligned} \int \sec^n x dx &= \int u dv \\ &= uv - \int v du \\ &= \sec^{n-2} x \tan x - (n-2)\int \tan^2 x \sec^{n-2} x dx \\ &= \sec^{n-2} x \tan x - (n-2)\int (\sec^2 - 1)\sec^{n-2} x dx \\ &= \sec^{n-2} x \tan x - (n-2)\left(\int \sec^n x dx - \int \sec^{n-2} x dx\right) \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x dx \end{aligned}$$

$$(n-1)\int \sec^n x dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x dx$$

$$\int \sec^n x dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x dx \quad \square$$

Derivative, Area and Integral Relationship

An intuitive explanation of the relationship between area, the derivative, and anti-derivative or integral is given below.

We are given a function *f* that is continuous on some interval. Let *F* describe the area of the region bounded between *f* and the *x*-axis over the interval.

Chose a point *x* and a nearby point *x* + *h* in the interval. The area of this region if then $F(x+h)-F(x)$.

This area can also be approximated as $f(x+h)\cdot h$ for small *h* so,

$$F(x+h)-F(x) \approx f(x+h)\cdot h$$
$$\frac{F(x+h)-F(x)}{h} \approx f(x+h).$$

If we take $\lim h \to 0$ the approximation becomes exact, the left side becomes the $F'(x)$ and the right side *f*(*x*): $F'(x) = f(x)$.

This shows that the derivative of the function that describes the area under a curve is the function that describes that curve.