# Geometric Transformations

Below a point $(x, y)$ on a **pre-image** is transformed to a new point $(x', y')$ on the **image**. All but dilation are isometric transformations.

## Translation

The image is shifted horizontally by *h* and vertically by *k*:

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|---|
| $(x', y') = (x+h, y+k)$ |

## Reflection on a Line

Points of the image have these three properties based on the line of reflection and the points of the pre-image:

1) Each image point lies on the line perpendicular to the line of reflection containing its corresponding pre-image point.
2) Each image point lies on the other side of the line of reflection as its corresponding pre-image point.
3) Each image point is the same distance from the line of reflection as its corresponding pre-image point.

Some common reflections formulas and the general formula are shown below.

| | | | |
|---|---|---|---|
| On *x*-axis, $y = 0$<br>$(x', y') = (x, -y)$ | On *y*-axis, $x = 0$<br>$(x', y') = (-x, y)$ | On $y = x$<br>$(x', y') = (y, x)$ | On $y = -x$<br>$(x', y') = (-y, -x)$ |
| On line $y = mx + b$: $(x', y') = \left( \frac{2my - 2mb - (m^2 - 1)x}{m^2 + 1}, \frac{2mx + 2b + (m^2 - 1)y}{m^2 + 1} \right)$ | | | On line $x = c$:<br>$(x', y') = (2c - x, y)$ |

## Reflection on a Point

Points of the image have these three properties based on the point of reflection and the points of the pre-image:

1) Each image point lies on the line containing the point of reflection and its corresponding pre-image point.
2) Each image point lies on the other side of the point of reflection as its corresponding pre-image point.
3) Each image point is the same distance from the point of reflection as its corresponding pre-image point.

Reflection on the origin is the most common point of reflection. This and the general case are shown below.

| | |
|---|---|
| On the origin, (0, 0): $(x', y') = (-x, -y)$ | On the point $(a, b)$: $(x', y') = (-x + 2a, -y + 2b)$ |

## Rotation on a Point

Points of the image have these two properties based on the point of rotation and the angle of rotation $\theta$:

1) Each image point is the same distance from the point of rotation as its corresponding pre-image point.
2) The angle formed by each image point, the point of rotation, and the corresponding pre-image point is $\theta$.

Rotations about the origin are most common. Some of these and the general case are shown below.

| | | | |
|---|---|---|---|
| On origin by 90°<br>$(x', y') = (-y, x)$ | On origin by 180°<br>$(x', y') = (-x, -y)$ | On origin by 270°<br>$(x', y') = (y, -x)$ | On origin by $\theta$<br>$(x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ |
| On point $(a, b)$ by $\theta$: $(x', y') = ((x-a)\cos\theta - (y-b)\sin\theta + a,\ (x-a)\sin\theta + (y-b)\cos\theta + b)$.<br>This is equivalent to 1) Translate the point of rotation and image to the origin, 2) Rotate the image about the origin, 3) Translate the image and the point of rotation together so the point of rotation is in its original position. | | | |

## Dilation

A **dilation** creates an image by specifying a **scale factor** and a point called the **scaling origin**. The pre-image and image of a dilation are similar, but not congruent, unless the scaling is $\pm 1$. Points of the image have these three properties:

1) Each image point lies on the line connecting its corresponding pre-image point and the scaling origin.
2) Each image point lies on the same side of the line as its corresponding pre-image point if the scale factor is positive and on the opposite side if negative.
3) The distance from the scaling origin to a point on the image is the product of the scale factor and the distance from the scaling origin to the corresponding pre-image point.

A general dilation with a scale factor *s* and scaling origin $(a, b)$ is show below.

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|---|
| $(x', y') = (sx + a(1-s),\ sy + b(1-s))$ |

## Transformation in Matrix Form

### *Translation*

By $h$ horizontally and $k$ vertically: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$

### *Reflection on a Line*

| | | | |
|---|---|---|---|
| On $x$-axis, $y = 0$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On $y$-axis, $x = 0$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On $y = x$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On $y = -x$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| On line $y = mx + b$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{m^2+1}\left(\begin{bmatrix} -(m^2-1) & 2m \\ 2m & (m^2-1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2mb \\ 2b \end{bmatrix}\right)$ | | On line $x = c$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2c \\ 0 \end{bmatrix}$ | |

### *Reflection on Point*

| | |
|---|---|
| On the origin: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On point $(a, b)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2a \\ 2b \end{bmatrix}$ |

### *Rotation*

| | | |
|---|---|---|
| On the origin by 90° $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On the origin by 180° $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On the origin by 270° $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ |
| On the origin by angle $\theta$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ | On point $(a, b)$ by angle $\theta$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b\sin\theta - a\cos\theta + a \\ -a\sin\theta - b\cos\theta + b \end{bmatrix}$ | |

### *Dilation*

By scale factor $s$ with scaling origin $(a, b)$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = s\begin{bmatrix} x \\ y \end{bmatrix} + (1-s)\begin{bmatrix} a \\ b \end{bmatrix}$

## Reflectional Symmetry

A **line of symmetry** divides an image into two halves such that the reflection of one half about the line of symmetry is identical to the other half. A figure may have zero, one, or more lines of symmetry. A figure is said to have **reflectional symmetry** if it has at least one line of symmetry. Examples:

- A scalene triangle has zero lines of symmetry.
- A non-equilateral isosceles triangle has one line of symmetry.
- A non-square rectangle has two lines of symmetry.
- An equilateral triangle has three lines of symmetry.
- A circle has an infinite number of lines of symmetry.

## Rotational Symmetry

The number of possible rotations less than 360° of an image which are indistinguishable from the pre-image is called the **order of rotational symmetry.** The order includes the original pre-image, so a rotational symmetry of order 1 means the pre-image has no rotational symmetry. If $\alpha$, $0° < \alpha \leq 360°$, is the smallest angle of rotation that gives an image indistinguishable from the pre-image then the order is $360°/\alpha$. Examples:

- A scalene triangle has order of rotational symmetry of 1.
- A non-square rectangle has order of rotational symmetry of 2.
- An equilateral triangle has order of rotational symmetry of 3.
- A square has order of rotational symmetry of 4.
- A circle has an infinite order of rotational symmetry.