Geometric and Arithmetic Sequence and Series

Sequence

A sequence is a function whose domain is the set of natural numbers. The natural numbers are the term numbers and the range of the function is the set of term values. Two common sequences are the arithmetic and geometric.

Arithmetic Sequence

$$t_n = t_1 + (n-1)d$$

- d is the common difference.
- Each term is the sum of the prior term and d.

Geometric Sequence

$$t_n = t_1 r^{n-1}$$

- r is the common ratio.
- Each term is the product of the prior term and r.

Arithmetic Means

To find the arithmetic mean between two numbers first find the common difference. For example, find the three arithmetic means between 13 and -7:

- Take $t_1 = 13$, t_2, t_3, t_4 as the three unknown arithmetic means and $t_5 = -7$ and so $n = 5$.
- Use the arithmetic sequence equation $t_n = t_1 + (n-1)d \;\rightarrow\; -7 = 13 + (5-1)d \;\rightarrow\; d = -5$.
- Use the arithmetic sequence equation to find t_2, t_3, t_4 which are the means: 8, 3 −2.

Geometric Means

To find the Geometric means between two numbers first find the common ratio. Pay attention for possible complex solutions. For example, find the three geometric means between 7 and 112:

- Take $t_1 = 7$, t_2, t_3, t_4 as the three unknown geometric means and $t_5 = 112$ and so $n = 5$.
- Use the geometric sequence equation $t_n = t_1 r^{n-1} \;\rightarrow\; 112 = 7r^{5-1} \;\rightarrow\; r^4 = 16$. The solution set for r is $\{\, 2, -2, 2i, -2i \,\}$.
- Use the geometric sequence equation to find t_2, t_3, t_4 for each solution of r.
- This gives the four sets of geometric means: 14, 28, 56; −14, 28, −56; $14i$, −28, $-56i$; $-14i$, −28, $56i$.

Series

A series is the sum of the terms of a sequence. The n^{th} partial sum S_n is the sum of the first n terms of a sequence. A series converges to a limit S if $\lim n \to \infty\, S_n = S$. Otherwise it diverges. The arithmetic series diverges. The geometric series converges for $|r| < 1$ and diverges otherwise.

Arithmetic Partial Sum

$$\begin{aligned} S_n &= \sum_{k=1}^{n} t_1 + (k-1)d \\ &= \frac{n}{2}\left(t_1 + t_n\right) \\ &= \frac{n}{2}\left(2t_1 + (n-1)d\right) \end{aligned}$$

Geometric Partial Sum

$$\begin{aligned} S_n &= \sum_{k=1}^{n} t_1 r^{k-1} \\ &= t_1 \frac{1-r^n}{1-r} \end{aligned}$$

$$S = \lim_{n\to\infty} S_n = t_1 \frac{1}{1-r}, \text{ for } |r| < 1.$$

Repeating Decimal as Fraction

The methods below use the repeating fraction $a = 12.23001030103\cdots$.

Geometric Series Method

1. Identify and separate the repeating and non-repeating portion: $a = 12.23 + .000\overline{0103}$.
2. Identify t_1: The part to the right of the decimal up to but no including the first repetition, $t_1 = .0000103$.
3. Identify r: $r = 10^{-n}$ where n is the number of digits in the repeating pattern, including zeros, $r = 10^{-4} = .0001$.
4. Use the infinite form of the geometric series and simplify:

$$a = 12.23 + \left(0.0000103\right)\frac{1}{1-.0001} = 12 + \frac{23}{100} + \left(\frac{103}{10000000}\right)\frac{10000}{9999} = 12 + \frac{23}{100} + \frac{103}{9999000} = 12\frac{2299873}{9999000}$$

Difference Method

1. First, multiply *a* by the factor of 10 needed to move the decimal place to the beginning of the 2nd repeat of the repeating pattern: $10^7 a = 122300103.\overline{0103}$.
2. Second, multiply *a* by the factor of 10 needed move the decimal place to the beginning of the 1st repeat of the repeating pattern: $10^3 a = 12230.\overline{0103}$
3. Subtract the second product from the first product. This removes the repeating decimal:

$$10^7 a - 10^3 a = 9999000a = \begin{array}{r} 122300103.\overline{0103} \\ - \quad 12230.\overline{0103} \\ \hline 122287873.0000. \end{array}$$

4. Solve for *a* and simplify: $a = \dfrac{122287873}{9999000} = 12\dfrac{2299873}{9999000}$.