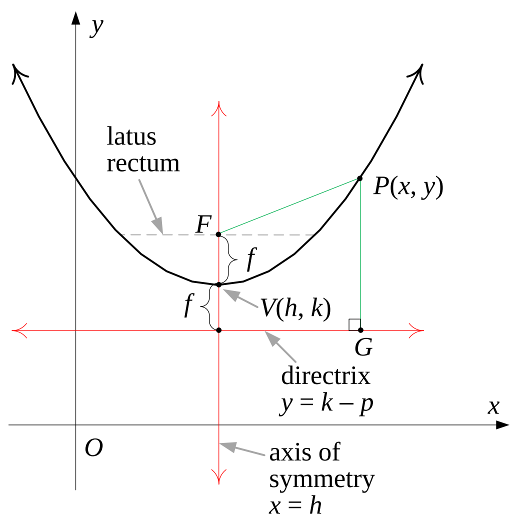
Parabola

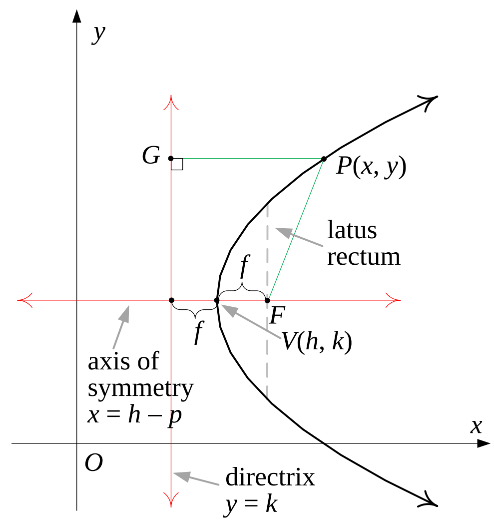
A parabola is the set of all points in a plane which are equidistant from a fixed line called the directrix and a fix point not on the line, called the focus.

The figures below show a vertical and horizontal opening parabola. The **axis of symmetry** passes through the **focus** *F* and **vertex** *V*. and is perpendicular to the **directrix**. The vertex lies on the parabola and is the midpoint of the line segment from the focus to the point of intersection of the axis of symmetry and directrix. For any point *P* on the parabola *FP* = *GP*, where *GP* is perpendicular to the directrix.

Vertical Parabola



Horizontal Parabola



From the definition of a parabola the length from the focus to the vertex must be the same as the distance from the vertex to the directrix. If we call this value $|f|$, then we can show that $f = 1/(4a)$ where *a* is the coefficient of the squared term of the parabola in standard polynomial form (also known as general form). *f* is positive if the parabola opens upward or to the right, negative otherwise.*

The **latus rectum** is the chord through the parabola's focus and parallel to the directrix. Its length is $4|f|$.

The standard form of a parabola (also known as vertex form) is derived from the standard polynomial form by completing the square. It is seen in two variations: In one the squared term has the coefficient *a*, which is just the coefficient of the squared term in standard polynomial form. In the other the coefficient is placed in front of the linear term and expressed in terms of *f* where $f = 1/(4a)$.*

Standard Parabola Form

$4f(y-k) = (x-h)^2$ or $y-k = a(x-h)^2$, $f = \dfrac{1}{4a}$

axis of symmetry: $x = h$
vertex: (h, k)
focus: $(h, k + f)$
directrix: $y = k - f$

Standard Parabola Form

$4f(x-h) = (y-k)^2$ or $x-h = a(y-k)^2$, $f = \dfrac{1}{4a}$

axis of symmetry: $y = k$
vertex: (h, k)
focus: $(h + f, k)$
directrix: $x = h - f$

Standard Polynomial Form: $y = ax^2 + bx + c,\ a \neq 0$

axis of symmetry: $x = \dfrac{-b}{2a}$

vertex: $\left(-b/2a,\ c - b^2/4a\right)$

focus: $\left(-b/2a,\ c - (b^2 - 1)/4a\right)$

directrix: $y = c - (b^2 + 1)/4a$

symmetric point for (x_1, y_1) is $(2h - x_1, y_1)$

Standard Polynomial Form: $x = ay^2 + by + c,\ a \neq 0$

axis of symmetry: $y = \dfrac{-b}{2a}$

vertex: $\left(c - b^2/4a, -b/2a\right)$

focus: $\left(c - (b^2 - 1)/4a, -b/2a\right)$

directrix: $x = c - (b^2 + 1)/4a$

symmetric point for (x_1, y_1) is $(x_1, 2k - y_1)$

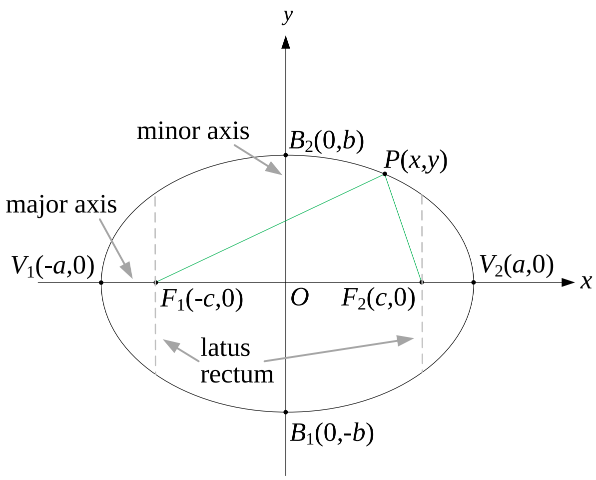
* [The symbol *a* or *p* is often used instead of *f*. Using *a* is confusing because it is usually used for the coefficient of the squared term of a quadratic in standard polynomial form, in which case $a = 1/(4f)$. Using *p* is confusing because this symbol is used in the polar form of conics where $p = 2f$.]

Ellipse

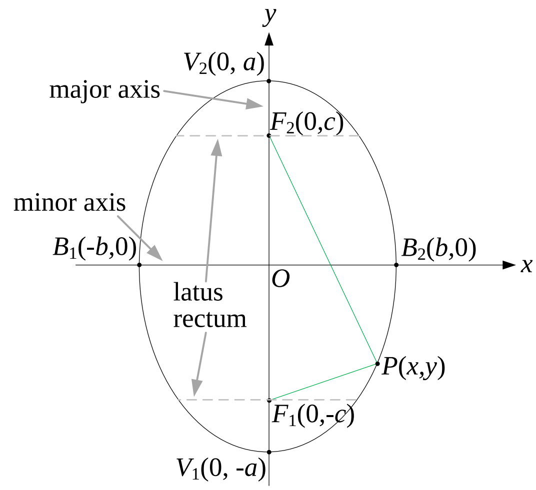
An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points called the foci is a constant.

The figures below show an ellipse with a horizontal major axis and another with a vertical major axis, both centered on the origin. The **major axis** is the line segment $\overline{V_1V_2}$ where V_1 and V_2 are called **vertices**. The **center** of the ellipse is the midpoint of the major axis. The **foci** are F_1 and F_2, and for any point P on the ellipse the distance $F_1P + F_2P$ is a constant. The **minor axis** is the line segment $\overline{B_1B_2}$ which also has its midpoint at the center. The major axis is distinguishable from the minor axis because it is the longer of the two. If the major and minor axes are equal the equation is that of a circle and the two foci merge to a single point at the center so that $a = b$ equals the radius.

Ellipse Major Axis Horizontal



Ellipse Major Axis Vertical



Horizontal	Common	Vertical
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ verticies: $(h\pm a,\ k)$ foci: $(h\pm c,\ k)$	center: (h,k) $c^2=a^2-b^2$ and $a>b$ minor axis length $=2b$ major axis length $=2a=F_1P+F_2P$	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$ verticies: $(h,\ k\pm a)$ foci: $(h,\ k\pm c)$

Summary

- The major axis is parallel to the axis of the variable of the term with the larger dominator, which by convention is a^2.
- The squared terms are **the same sign** (in a hyperbola one is positive and one is negative).
- The foci are closer to the center than the vertices (a hyperbola has foci farther from center than vertices).
  - The distance from the center to the vertex is a.
  - The distance from the center to the foci is $c=\sqrt{a^2-b^2}$.
- A circle is an ellipse with $a = b$.
- A **latus rectum** is a chord through a focus and is parallel to the minor axis.

Eccentricity

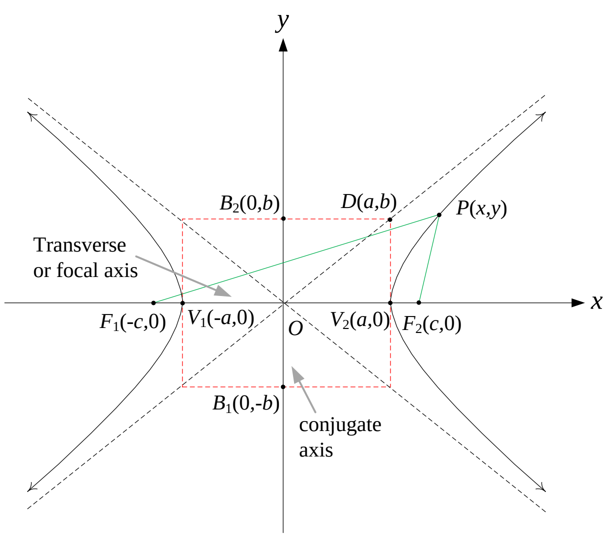
The **eccentricity *e*** of an ellipse is defined as $e = c/a$. Since $0 < c < a$ for an ellipse this means $0 < e < 1$. The eccentricity is a measure of the "roundness" vs. "flatness" of the ellipse. The smaller e is the closer together the foci are compared to distance between the vertices. In the limit as $e \to 0$ the ellipse approaches a circle with radius a. At the other extreme as $e \to 1$ the ellipse approaches a line segment of length $2a$.

Hyperbola

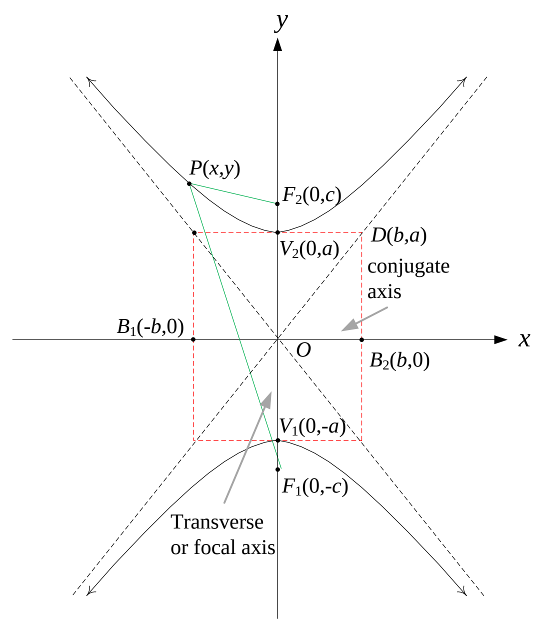
A hyperbola is the set of all points in a plane for which the absolute value of the difference of the distance from two fixed points not on the hyperbola, called the foci, is a constant.

The figures below show a horizontal and vertical opening hyperbola. A hyperbola forms two disjoint branches and has two focal points, two vertex points and a center as in an ellipse. The foci are farther from the center than the vertices (the opposite of an ellipse). The center of the hyperbola is the midpoint of the **transverse axis** which is the segment $\overline{V_1V_2}$ joining the vertices. The foci are labeled F_1 and F_2 and $|F_1P - F_2P|$ is a constant for all points P on the hyperbola. The **conjugate axis** is the line segment $\overline{B_1B_2}$. The branches of the hyperbola asymptote to lines that intersect the center and the points $(\pm a, \pm b)$ as shown below.

Horizontal Hyperbola



Vertical Hyperbola



Horizontal	Common	Vertical
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ vertices: $(h \pm a, k)$ foci: $(h \pm c, k)$ asymptote: $y - k = \pm\frac{b}{a}(x-h)$	center: (h, k) $c^2 = a^2 + b^2$ and $c > a$ transverse axis length = $2a$ = $\lvert F_1P - F_2P \rvert$ conjugate axis length = $2b$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ vertices: $(h, k \pm a)$ foci: $(h, k \pm c)$ asymptote: $y - k = \pm\frac{a}{b}(x-h)$

Summary

- By convention the positive term uses a^2 is in its denominator.
- The branches open in the direction parallel to the axis of the variable with the positive term.
- The squared terms are of **opposite** sign (in an ellipse they are the same positive).
- The foci are farther from the center than the vertices (an ellipse has foci closer to the center than the vertices).
  - The distance from the center to the vertex is a.
  - The distance from the center to the foci is $c = \sqrt{a^2 + b^2}$.
- The equation of the asymptote is the same as that of the hyperbola except with the right side 1 replaced with 0.
- The **number** a is called the **semifocal axis**.
- The **number** b is called the **semiconjugate axis**.
- A **latus rectum** is a chord through a focus and parallel to the conjugate axis.

Rotated Conic

Type of Conic

The polynomial $Ax^2 + Cy^2 + Dx + Ey + F = 0$, with *A* and *C* not both zero, can be put into the standard form of a conic by completing the square for *x* and *y*. If *A* or *C* is zero, then there is only one squared variable and the polynomial represents a parabola. If both *A* and *C* are the same sign, this gives a sum of perfect squares which represents an ellipse. If *A* and *C* are of opposite sign this produces a difference of perfect squares which represents a hyperbola. This is summarized below.

$AC < 0$	Hyperbola
$AC > 0$ • $A = C$	Ellipse • Circle
$AC = 0$	Parabola

Rotation of Coordinate System

Given an *xy* coordinate system, if we introduce a second coordinate system $x'y'$ which is rotated about the origin of the original system by a positive acute angle θ we can relate the coordinates of a point *P*, with original coordinates (x, y), to its coordinates (x', y') in the rotated coordinate system. This is done below:

From the figure to the right we can write the coordinates as,

$$x' = r\cos\alpha \quad \text{and} \quad y' = r\sin\alpha \qquad (1)$$
$$x = r\cos(\alpha+\theta) \quad \text{and} \quad y = r\sin(\alpha+\theta) \qquad (2).$$

Using the double angle identities (2) can be expanded as,

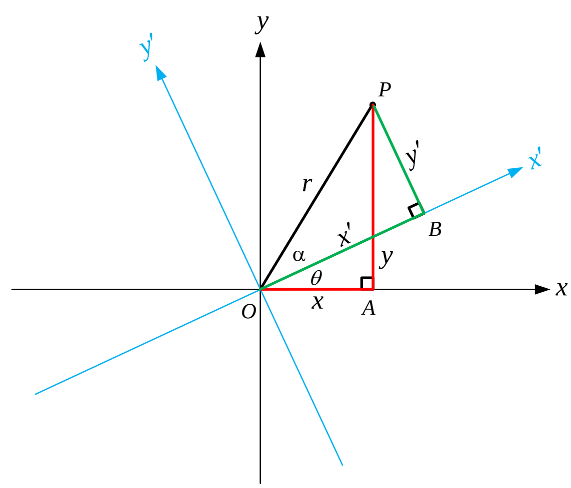
$$x = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \qquad y = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta.$$

Substituting (1) into the above we have:

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= y'\cos\theta + x'\sin\theta. \end{aligned} \qquad (3)$$

With a little algebra, we can solve for x' and y':

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= y\cos\theta - x\sin\theta. \end{aligned} \qquad (4)$$



Discriminant Test

The polynomial given at the beginning of this section did not include the "cross term" *Bxy*. If this is included in the polynomial, its effect is to create a conic rotated from the horizontal or vertical (but not necessarily the same type of conic as if *B* were zero).

If we have the polynomial equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ we can express this in terms of $x'y'$ by substituting (3) for *x* and *y*. After collecting all like terms the result is a polynomial in x', y' with coefficients expressed in terms of θ. Using prime marks to denote these new coefficients we can write $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F'$. To determine the rotation that eliminates B' write $B' = 0$ and solve for the angle θ. Doing this gives:

Angle of Coordinate rotation that eliminates B'	$\theta = \frac{1}{2}\cot^{-1}\frac{A-C}{B}$

You can also show that $B^2 - 4AC = B'^2 - 4A'C'$. By using the angle above to eliminate B' you then have $B^2 - 4AC = -4A'C'$. This allows us to look at $B^2 - 4AC$, which is called the discriminant, to determine the type of conic:

$B^2 - 4AC > 0$	Hyperbola
$B^2 - 4AC < 0$ • $A = C,\ B = 0$	Ellipse • Circle
$B^2 - 4AC = 0$	Parabola

Notice there is no need to know what angle eliminates B' to use the discriminant test. It is sufficient to know that such an angle exists that allows $B^2 - 4AC = -4A'C'$.

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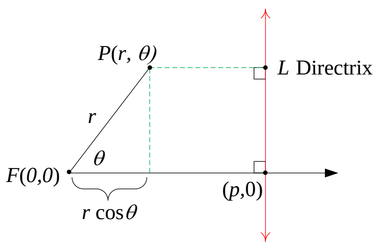
Conics in Polar Coordinates

Equation for Conics

The conic sections: ellipse, parabola, and hyperbola, can be defined by a single geometric definition:

Let *L* be a line called the directrix and *F* be a point not on *L* called the focus. A conic section is the set of points *P* in the plane such that PF/PL is a positive constant called the eccentricity and given the symbol *e*.

The value of e determines if the set of points is an ellipse, parabola or hyperbola and is a generalization of the eccentricity of the ellipse. Below is a diagram illustrating the definition and deriving the equation of the conic in a polar coordinate system, assuming a vertical directrix p units to the right of the pole.



From the figure $e = \dfrac{PF}{PL} = \dfrac{r}{p - r\cos\theta}$. Solving for r gives

$$r = \frac{ep}{1 + e\cos\theta}, \quad e > 0$$

The four combinations of positive or negative p and vertical or horizontal directrix are summarize below:

Vertical directrix

Directrix is $\perp$ to polar axis at a distance p units to **left** of the pole.	$r = \dfrac{ep}{1 - e\cos\theta}$
Directrix is $\perp$ to polar axis at a distance p units to **right** of the pole.	$r = \dfrac{ep}{1 + e\cos\theta}$

Horizontal directrix

Directrix is ‖ to polar axis at a distance p units **above** of the pole.	$r = \dfrac{ep}{1 + e\sin\theta}$
Directrix is ‖ to polar axis at a distance p units **below** the pole.	$r = \dfrac{ep}{1 - e\sin\theta}$

Conic Type Based on *e*

Recall that for the ellipse and hyperbola c is the distance from the center to a focus and a is the distance from the center to a vertex. The eccentricity is the ratio of c and a:

$$e = \frac{c}{a}$$

For an ellipse, the focus is closer to the center than the vertex and vice versa for a hyperbola, thus $e < 1$ for an ellipse and $e > 1$ for a hyperbola. A parabola is the border case between a hyperbola and ellipse where e = 1:

$e < 1 \rightarrow$ the conic is an ellipse.
$e = 1 \rightarrow$ the conic is a parabola.
$e > 1 \rightarrow$ the conic is a hyperbola.

While a parabola does not have a center or second focus, we can think of a parabola as an ellipse with its second focus and its center an infinite distance from its first focus, along the major axis. In this way, we can imagine that the difference between c and a, which remains constant, becomes infinitesimally small in comparison to both c and a so that $e = c/a \rightarrow 1$ for a parabola. However, the absolute difference $a - c$ does not approach zero but remains constant and, being the distance from the vertex to the focus, is the parameter f used in the standard form of a parabola in a rectangular coordinate system.