Common Polynomial Factors

Common Factors and Their Expansions

Factored Form = Expanded Form	Name
$(a+b)^2 = a^2 + 2ab + b^2$	Square of Sum
$(a-b)^2 = a^2 - 2ab + b^2$	Square of Difference
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	Cube of Sum
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	Cube of Difference
$(a+b)(a-b) = a^2 - b^2$	Difference of Squares
$(a-b)(a^2+ab+b^2) = a^3 - b^3$	Difference of Cubes
$(a+b)(a^2-ab+b^2) = a^3 + b^3$	Sum of Cubes

Sum and Difference of Powers

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}) = (a-b)\sum_{i=1}^{n} a^{n-i}b^{i-1}$	for odd integer *n*. *
$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + a^2b^{n-3} - ab^{n-2} + b^{n-1}) = (a+b)\sum_{i=1}^{n} (-1)^{i-1} a^{n-i}b^{i-1}$	for odd integer *n*.

* The difference of powers is also valid for even positive integer exponents. However, it is not as effective in factoring as treating such cases as the differences of squares, $\left(a^{n/2}\right)^2 - \left(b^{n/2}\right)^2$ repeatedly until reaching $a-b$.

Examples

$a^6 + b^3 = \left(a^2\right)^3 + b^3$ $= \left(\left(a^2\right)+b\right)\left(\left(a^2\right)^2 - a^2b + b^2\right)$ $= \left(a^2+b\right)\left(a^4 - a^2b + b^2\right)$	$5a^5 + 5000a^2 = 5a^2\left(a^3 + 1000\right)$ $= 5a^2\left(a^3 + 10^3\right)$ $= 5a^2\left(a+10\right)\left(a^2 - 10a + 100\right)$
$a^6 - b^6 = \left(a^3\right)^2 - \left(b^3\right)^2$ $= \left(a^3 - b^3\right)\left(a^3 + b^3\right)$ $= \left(a-b\right)\left(a^2+ab+b^2\right)\left(a+b\right)\left(a^2-ab+b^2\right)$ If we fist formed the difference of cubes, we need to do more work to get the same result, $a^6 - b^6 = \left(a^2\right)^3 - \left(b^2\right)^3$ $= \left(a^2 - b^2\right)\left(a^4 + a^2b^2 + b^4\right)$ $= \left(a-b\right)\left(a+b\right)\left(a^4 + a^2b^2 + b^4\right).$ To factor the right product the method of "Factoring Quartics by Completing the Square" is needed. See last example in that section.	$a^2 + 2b - b^2 - 1 = a^2 - \left(b^2 - 2b + 1\right)$ $= a^2 - \left(b-1\right)^2$ $= \left(a - (b-1)\right)\left(a + (b-1)\right)$ $= \left(a-b+1\right)\left(a+b-1\right)$ $1024x^{10} + y^5 = \left(\left(2x\right)^2\right)^5 + y^5$ $= \left(4x^2 + y\right)\left(\left(4x^2\right)^4 - \left(4x^2\right)^3 y + \left(4x^2\right)^2 y^2 - 4x^2y^3 + y^4\right)$ $= \left(4x^2 + y\right)\left(256x^8 - 64x^6y + 16x^4y^2 - 4x^2y^3 + y^4\right)$

Factoring by Grouping

Look for groupings that allow a common factor to be identified. Try looking for two pairs of coefficients with the same ratio, the pairs *might* form a group that allows factoring.

Examples

$3xyz-4yz^2+15xy^2-20y^2z=(3xyz+15xy^2)-(4yz^2+20y^2z)$ $=3xy(z+5y)-4yz(z+5y)$ $=(z+5y)(3xy-4yz)$ $=y(z+5y)(3x-4z).$	$ab-bc-c^2+ac=(ab-bc)-(c^2-ac)$ $=b(a-c)-c(c-a)$ $=b(a-c)+c(a-c)$ $=(a-c)(b+c)$
$5c^3+2c^2-5cd^2-2d^2=c^2(5c+2)-d^2(5c+2)$ $=(c^2-d^2)(5c+2)$ $=(c-d)(c+d)(5c+2)$	$4a^3c^2-ab^2c^2+4a^2bc^2-b^3c^2-36a^3d^2+9ab^2d^2-36a^2bd^2+9b^3d^2$ $=c^2(4a^3-ab^2+4a^2b-b^3)-9d^2(4a^3-ab^2+4a^2b-b^3)$ $=(c^2-9d^2)(4a^3-ab^2+4a^2b-b^3)$ $=(c-3d)(c+3d)(a(4a^2-b^2)+b(4a^2-b^2))$ $=(c-3d)(c+3d)(a+b)(4a^2-b^2)$ $=(c-3d)(c+3d)(a+b)(2a-b)(2a+b)$

Factoring Quartics by Completing the Square

If a polynomial can be expressed in the form $ax^{4n}+bx^{2n}+c$ and does not factor as a quadratic in form, under some special conditions it may be possible to complete the square to form a difference of squares. The conditions required to do this are,

- Both a and c are perfect squares.
- A term d exists such that $b-d$ is a perfect square so we can add and subtract dx^{2n} from the polynomial to form the difference of two perfect squares, $(ax^{4n}+dx^{2n}+c)-(d-b)x^{2n}$. Completing the square of the first three terms above gives $d=\pm 2\sqrt{ac}$.

Examples

$x^4-19x^2+25=x^4-2\sqrt{25}x^2+25-19x^2+2\sqrt{25}x^2$ $=x^4-10x^2+25-(19-10)x^2$ $=x^4-10x^2+25-9x^2$ $=(x^2-5)^2-(3x)^2$ $=(x^2-3x-5)(x^2+3x-5)$	$x^4+4=x^4+2\sqrt{4}x^2+4-2\sqrt{4}x^2$ $=x^4+4x^2+4-4x^2$ $=(x^2+2)^2-(2x)^2$ $=((x^2+2)-(2x))((x^2+2)+(2x))$ $=(x^2-2x+2)(x^2+2x+2)$
$x^4+6x^2+25=x^4+2\sqrt{25}x^2+25+6x^2-2\sqrt{25}x^2$ $=x^4+10x^2+25-(10-6)x^2$ $=x^4+10x^2+25-4x^2$ $=(x^2+5)^2-(2x)^2$ $=((x^2+5)-2x)((x^2+5)+2x)$ $=(x^2-2x+5)(x^2+2x+5)$	$4x^4-x^2+36=4x^4+2\sqrt{144}x^2+36-x^2-2\sqrt{144}x^2$ $=4x^4+24x^2+36-25x^2$ $=4(x^4+6x^2+9)-(5x)^2$ $=(2(x^2+3))^2-(5x)^2$ $=(2(x^2+3)-5x)(2(x^2+3)+5x)$ $=(2x^2-5x+6)(2x^2+5x+6)$
$a^4+a^2b^2+b^4=a^4+2a^2b^2+b^4-2a^2b^2+a^2b^2$ $=(a^4+2a^2b^2+b^4)-(a^2b^2)$ $=(a^2+b^2)^2-(ab)^2$ $=(a^2+ab+b^2)(a^2-ab+b^2)$	