Important Definitions and Theorems

Definitions

Limit

Given a function *f* defined in an open interval containing the number *a*, except that *f* need not be defined at *a*, if for any number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ when } 0 < |x - a| < \delta ,$$

then the limit of *f* as *x* approaches *a* equals *L*, written $\lim_{x \to a} f(x) = L$.

Continuity

A function *f* is continuous at *c* if all the following are true:

1) $f(c)$ is defined
2) $\lim_{x \to c} f(x)$ exists
3) $\lim_{x \to c} f(x) = f(c)$.

Increasing, Decreasing and Constant Functions

1) *f* is **increasing*** on an interval *I* if $f(b) > f(a)$ for all *a* and *b* in *I* where $b > a$.
2) *f* is **decreasing*** on an interval *I* if $f(b) < f(a)$ for all *a* and *b* in *I* where $b > a$.
3) *f* is **constant** on an interval *I* if $f(a) = f(b)$ for all *a* and *b* in *I*.

* Some texts call 1 and 2 above strictly increasing and strictly decreasing. Without this qualifier they use $f(b) \geq f(a)$ for 1 and $f(b) \leq f(a)$ for 2. With this definition a constant function is both increasing and decreasing.

Local (Relative) Extrema

1) f has a **local maximum*** at *c* if there exists an open interval *I* containing *c* where $f(c) \geq f(x)$ for all *x* in *I*.
2) f has a **local minimum*** at *c* if there exists an open interval *I* containing *c* where $f(c) \leq f(x)$ for all *x* in *I*.

* Some authors allow *c* to be the endpoint of a closed domain. In this case *I* is a half-open interval. This allows endpoints to be candidates for local extrema which are excluded in the definition above by the requirement that *I* to be an open interval.

Absolute (Global) Extrema

1) f has an **absolute maximum** at *c* in interval *I* if $f(c) \geq f(x)$ for all *x* in *I*.
2) f has an **absolute minimum** at *c* in interval *I* if $f(c) \leq f(x)$ for all *x* in *I*.

Concavity

If *f* is differentiable on an open interval *I*, then

1) *f* is **concave up** on *I* if f' is increasing on *I*.
2) *f* is **concave down** on *I* if f' is decreasing on *I*.

Point of Inflection

If *f* is continuous on an open interval containing *c* and if *f* changes the direction of its concavity at point $(c, f(c))$ then *f* is said to have an **inflection point at c** and the point $(c, f(c))$ on the graph of *f* is an **inflection point** of *f*.

- If the point occurs where the derivative is infinite it is called a **vertical inflection point**.
- If the point occurs where the derivative is zero it is called a **stationary inflection point** or **horizonal inflection point.**

Cusps and Corners

Cusps and **corners** occur on a *continuous* interval at a point where the first derivative does not exist *and* the left and right sided derivatives are distinct.

Theorems

Intermediate Value Theorem
If function f is *continuous* on an interval [a, b] and $f(a) \le L \le f(b)$ then there exists a value c in [a, b] such that $f(c) = L$.
In other words, f will take on every value between $f(a)$ and $f(b)$ over the interval [a, b].

Extreme Value Theorem
If function f is continuous on an interval [a, b] then there exists both an absolute maximum and absolute minimum of f on [a, b].

Rolle's Theorem
If function f is continuous on an interval [a, b] and differentiable on the interval (a, b) and $f(a) = f(b) = 0$ then there exists a value c in (a, b) such that $f'(c) = 0$.

Mean-Value Theorem
If function f is continuous on an interval [a, b] and differentiable on the interval (a, b) then there exists a value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Mean-Value Theorem for Integrals
If function f is continuous on an interval [a, b] then there exists a value c in (a, b) such

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \triangleq f_{av}.$$

The value f_{av} is defined as the average value of the function f on the interval [a, b].

Fundamental Theorem of Calculus

i. If function f is continuous on [a, b] and F is any antiderivative of f on [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$.

ii. If function f is continuous on [a, b] then for any constant c in [a, b] the function $F(x) \triangleq \int_c^x f(t)dt \;\; x \in [a,b]$ is an antiderivative of f and F is continuous on [a, b], differentiable on (a, b) and $F'(x) = f(x)$.

A geometric explanation of the relationship between area, the derivative, and anti-derivative (integral): Given a function f that is continuous on some interval, let F describe the area of the region bounded between f and the x-axis over the interval. Chose a point x and a nearby point $x + h$ in the interval. The area of this region if then $F(x+h) - F(x)$. This area can also be approximated as $f(x+h)\cdot h$ for small h so,

$$F(x+h) - F(x) \approx f(x+h)\cdot h$$
$$\frac{F(x+h) - F(x)}{h} \approx f(x+h).$$

If we take $\lim h \to 0$ the approximation becomes exact, the left side becomes the $F'(x)$ and the right side $f(x)$, thus $F'(x) = f(x)$. This shows that the derivative of the function that describes the area under a curve is the function that describes that curve.

One-to-One Theorem
If the domain of function f is an open interval and its derivative is either always positive or always negative, then f is one-to-one and it follows that its inverse is differentiable for all values in the *range* of f (which is the domain of f^{-1}).

Derivative of Inverse Function Theorem
Given a function f that has an inverse $f^{-1} = g$, if f is differentiable at $g(x)$ and $f'(g(x)) \ne 0$ then $g'(x) = \dfrac{1}{f'(g(x))}$.

If we call $y = f^{-1}(x) \;\to\; x = f(y)$ this can be expressed as $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$.

In other words, the slopes of the graphs of f and f^{-1} are multiplicative inverses (where they are both defined). To find the slope of f^{-1} at a solve $f(b) = a$ for b and then the slope is $\left(f^{-1}\right)'(a) = \dfrac{1}{f'(b)}$, where $a = f(b)$.