Binomial Theorem

The **Binomial Theorem** relates a binomial raised to a non-negative integer power to a series named the binomial series.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \text{ , } n \text{ is a non-negative integer,}$$

and $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is called the binomial coefficient; it gives the coefficients in the expansion of a binomial raised to a power, e.g., $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

term #	1	2	3	4	5	6	7	8

Binomial Coefficient Related Facts

- The term number is one more than the ascending exponent → there is one more term than the degree of the binomial.
- The next coefficient can be found from the current coefficient by dividing the product of the current coefficient and its associated descending exponent by the current term number, e.g., term 4 coefficient above is $21 \cdot 5/3 = 35$.
- The coefficients are symmetric about the center term(s).
- The coefficients sum to 2^n. This can be shown by setting a and b equal to 1.
- The coefficients form Pascals Triangle where each entry is the sum of the nearest two entries above it:

1	$(a+b)^0$
1 1	$(a+b)^1$
1 2 1	$(a+b)^2$
1 3 3 1	$(a+b)^3$
1 4 6 4 1	$(a+b)^4$
1 5 10 10 5 1	$(a+b)^5$
1 6 15 20 15 6 1	$(a+b)^6$
1 7 21 35 35 21 7 1	$(a+b)^7$

- The coefficient is how many times the product $a^{n-k}b^k$ appears in the expansion of $(a+b)^n$ before summing like terms. This is the same as the enumeration of the ways k items can be chosen, without regard to order and without replacing items, from n distinct items (also the same as $n-k$ items chosen out of n because of the symmetry of the binomial coefficient).

Finding Terms in a Binomial Expansion

Example: Given the binomial $(2x^3 - 5y)^{13}$ find a) the term containing x^{27} and its term number; b) The 8th term; c) the term containing x^8.

a) i) First identify the a and b in the notation $(a+b)^n$: $a = 2x^3$ and $b = -5y$.

ii) Then determine the exponent needed by a: $27/3 = 9$.

$$\binom{13}{13-9} a^9 b^{13-9} = \binom{13}{4} a^9 b^4 = \frac{13!}{4!9!}(2x^3)^9(-5y)^4 = 715(2^9x^{27})(-5)^4 y^4 = 228800000x^{27}y^4$$

iii) The term number is one more than the exponent of b so this is the 5th term.

b) The term numbers ascend from 1 while the b exponent ascends from 0 so the 8th term contains b^7 and so it also contains $a^{13-7} = a^6$.

$$\binom{13}{7} a^6 b^7 = \frac{13!}{7!6!}(2x^3)^6(-5y)^7 = 1716(64x^{18})(-78125y^7) = -8580000000x^{18}y^7$$

c) The exponent needed by a is $8/3 = 2.\overline{33}$. Since this is not an integer and the binomial expansion defined above only has non-negative integer exponents there cannot be a term containing x^8.